# UNIT 3

*Exponential and Logarithmic Functions*

In this Unit, we will extend upon the basic knowledge that you likely possess regarding exponents and logarithms. We assume that you recall the basic exponent and logarithm properties, but these properties are summarized here.

## Lesson 3.0: Exponent Properties

**EXERCISE:** Use the supplied properties to fill in the table, where $a$ and $b$ are non-zero real numbers, and $m$ and $n$ are positive integers.

| | Rule | Example | Solution |
|---|---|---|---|
| 1 | $a^m \cdot a^n = a^{m+n}$ | $x^2 \cdot x^5$ | |
| 2 | $(ab)^m = a^m \cdot b^m$ | $(xy)^3$ | |
| 3 | $(a^m)^n = a^{mn}$ | $(x^2)^3$ | |
| 4 | If $m > n$, then $\frac{a^m}{a^n} = a^{m-n}$ | $\frac{x^5}{x^3}$ | |
| 5 | If $n > m$, then $\frac{a^m}{a^n} = \frac{1}{a^{n-m}}$ | $\frac{x^3}{x^5}$ | |
| 6 | $\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}$ | $\left(\frac{x}{y}\right)^3$ | |
| 7 | $a^{-m} = \frac{1}{a^m}$ | $3^{-2}$ | |
| 8 | $\frac{1}{a^{-m}} = a^m$ | $\frac{1}{2^{-3}}$ | |
| 9 | $a^0 = 1$, as long as $a \neq 0$. $0^0$ is undefined. | $5^0$ | |

Although there are many kinds of exponential functions that describe a wide variety of real-life situations, the most basic of these are of the form $f(x) = a \cdot b^x$ as you saw in the first Unit. Generally, we place two restrictions on the value of $b$. The first is that $b > 0$ and the second is that $b \neq 1$. Can you explain why each of these restrictions is necessary?

**Logarithm Properties**

For $b > 0, b \neq 1, M > 0, N > 0$, and natural number $k$:

| |
|---|
| 1. $\log_b(M) + \log_b(N) = \log_b(MN)$ |
| 2. $\log_b(M) - \log_b(N) = \log_b\left(\frac{M}{N}\right)$ |
| 3. $\log_b\left(M^k\right) = k\log_b M$ |
| 4. $\log_a x = \frac{\log_b x}{\log_b a}$<br>Change of Base Formula |

**EXTENSION** The logarithm properties follow directly from the laws of exponents. Can you prove this? For example, use Exponential Rule 1 to prove that Logarithm Rule 1 is true.

### REVIEW Exploration 3.0.1: Exponent and Logarithm Properties

Simplify the following expressions

**1.** $x^3 \cdot x^4$

**2.** $y^2 \cdot y^5$

**3.** $\dfrac{x^6}{x^2}$

**4.** $\left(a^3b^5\right)/\left(ac^3\right)$

**5.** $\left(2^2\right)^3$

**6.** $\left(w^4\right)^2$

**7.** $\left(4x^2y\right)^3$

**8.** $\left(5x^2y^7\right)^3\left(4y^8\right)$

**9.** $\left(9^{54}\right)^{10}/\left(9^{49}\right)^{11}$

Evaluate without a calculator. Recall that $y = \log_b x$ if and only if $b^y = x$.

**10.** $\log_5 25$

**11.** $\log_2 16$

**12.** $\log_{10} 1{,}000{,}000$

**13.** $\log_3 1$

**14.** $\log_{16} 4$

Solve for x.

**15.** $\log_3 x = -4$

**16.** $\log_{1/2} x = 8$

**17.** $\log_x 4 = 2/3$

**18.** $\log_x 16 = -4$

Evaluate without a calculator.

**19.** $\log_4 3 + \log_4 8$

**20.** $\log_2 225 - \log_2 5 + \log_2 3$

**21.** $4 \log_2 3$

**22.** $4 \log_8 3 - \log_8 6$

# Lesson 3.1: A Special Number

## Exploration 3.1.1: A Number Between 2 and 3

In this activity we will investigate an accumulation function related to the *algebraic measure* of the area under the curve of the function. By *algebraic measure*, we are referring to the measure of the area between the curve and the $x$-axis on a given interval.

$$f(t) = \frac{1}{t}$$

We will restrict our activities to the interval $t \in [1,3]$.

**1.** On graph paper, create coordinate axes with a scale of $0.1$ unit on each axis. The values on the $f(t)$ axis should range from $0$ to $1.1$, and the values on the $t$-axis should range from $0$ to $3.1$. Plot the graph of $f(t)$ on the domain interval $[1, 3]$ by plotting at least 8 evenly spaced values on the interval. Carefully draw a continuous curve through the values.

**2.** The function that represents the accumulated area under $f(t)$ on the interval $[1, x]$ where $x \in [1,3]$ will be called $L(x)$. What is the value of $L(1)$?

**3.** With a scale of $0.1$ on each of your axes, what is the value represented by the area of each square of the grid on your graph paper? Use this fact to approximate the value of $L(2)$.

**4.** Approximate the value of $L(3)$.

**5.** Use what you have discovered so far to estimate the value for $x \in [1,3]$ such that $L(x) = 1$. Do you know the special name given to this value?

**Research Extension:**

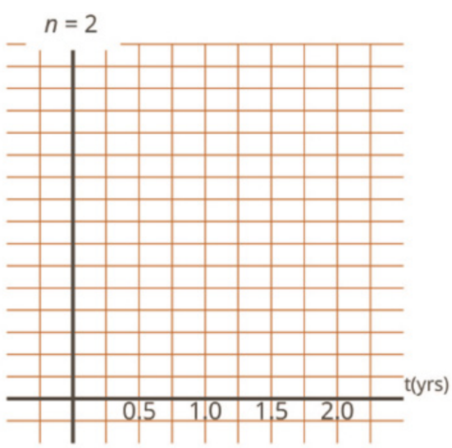
**6.** Since $L(x)$ is a *continuous* function, there is a theorem from calculus that guarantees that there is a unique value $x \in [1,3]$ such that $L(x) = 1$. Research this theorem and explain how it applies to this problem.

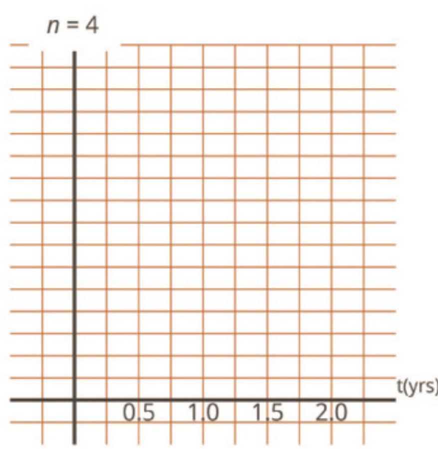
### Exploration 3.1.2: More on that Special Number

**1.** Write a generic exponential growth function $A(t)$ that has initial value $P$ and an annual growth rate (expressed as a decimal) of *r*.

**2.** The function that you wrote in part 1 describes compounding the growth once per year. Adjust this function to compound the growth *n* times per year. For example, if $r = 0.12\ (12\%)$ once per year, letting $n = 2$ would mean that the value of $A$ increases by $6\%$ each of $2$ times per year.

**3.** Your function from part 2 is called the compound interest function. Strictly speaking, this is not a continuous function. For example, if $12\%$ interest is compounded twice a year, then the value of the investment would only take on 3 discrete values during that year — the original value, $106\%$ of the original value, and $106\%$ of $106\%$ of the original value. Sketch graphs of the compound interest function over a two-year period for $n = 2$ and $n = 4$. You can let $P$ take on some arbitrary value, such as $100$.



**4.** Describe at least two things that happen to the "bars" on the graph as $n$ increases.

**5.** Describe how the appearance of the graph would change as you let $n$ approach infinity.

**6.** Let's define a quantity $x = \frac{n}{r}$. For a fixed value of $r$, what happens to $x$ as $n$ approaches infinity?

**7.** Now go back to your function from part 2. If $x = \frac{n}{r}$, then we can replace $\frac{r}{n}$ with ___________, and we can replace the $n$ in the exponent with ________________. Make these substitutions and write the revised formula here.

**8.** Your formula in part 7 should contain the expression $\left(1+\frac{1}{x}\right)^x$. In a graphing calculator, enter this expression under $Y1$, go to TBLSET (2nd WINDOW), and change the independent variable from Automatic to Ask. Now when you go to the TABLE, you can type in the values of $x$ that you want to substitute into the function instead of having the calculator pick them for you. Now use the Table to evaluate this expression for $x = 10, 100, 1000, 10000$, etc. Describe what happens to this expression as $x$ gets very large.

**9.** The number that this expression approaches as $x$ gets very large is called $e$, an irrational number that is approximately equal to ________________. Substitute *e* into your function from part 7. Describe the difference between what this function describes and what your function from part 2 describes.

**Historical Note:**

The mathematician John Napier (circa 1600) is credited as being the first to introduce the number $e$. Napier alluded to $e$ as a "special number" associated with his development of the theory of logarithms. It is Leonard Euler (circa 1720), however, that defined and used the symbol $e$ to refer to Napier's "special number." Euler also discovered many of this number's special properties. It is likely that Euler chose the symbol $e$ for this *natural number* as a reference to the "exponential."

!! Before proceeding to the next Lesson, it would be good to recall (as seen in Exploration 3.0.1) that the logarithm function is the inverse of the exponential function in such a way that if

$$\text{base}^{\text{exponent}} = \text{answer},$$

then

$$\log_{\text{base}}(\text{answer}) = \text{exponent}.$$

Thus, if $10^2 = 100$, then $\log_{10}(100) = 2$.

## Lesson 3.2: The Natural Logarithm Function as the Inverse of $e^x$

The irrational number $e$ is used extensively in the mathematics associated with finance and economics. As you have discovered, the number can be derived from exploring a special case of the amortization formula

$$A = P\left(1+\frac{r}{n}\right)^{nt},$$

where $A$ is the total value of an investment after $t$ years, $P$ is the initial or *principal* investment, $r$ is the rate of interest, and $n$ is the number of times that the interest is compounded per year. You are encouraged to research the connection between *e* and the mathematics of finance.

The next Exploration considers the inverse of the exponential function $y = f(x) = e^x$. For the function $f$, the symbol $f^{-1}$ will be used to denote the inverse of $f$. Recall, from Unit 1 that if we consider $f(x) = y$, then the inverse function is written as $f^{-1}(y) = x$. In the case of the exponential function, we can write $f^{-1}(y) = f^{-1}(e^x) = x$.

## Exploration 3.2.1: Logarithms in Carbon Dating

### Part A: Background

**1.** Explain how you know that the exponential function has an inverse.

**2.** Knowing the properties of the exponential function can help us decide on some properties that should be true of its inverse. Write $e^a = l$ and $e^b = m$. Consider the property $e^a e^b = e^{a+b}$. Use this to show that $f^{-1}(lm) = f^{-1}(l) + f^{-1}(m)$ is a property of the inverse function.

The inverse function we've been exploring is called the *natural logarithm*, and is written as $ln(y)$ . Rather than writing $f^{-1}(y) = x$, we can write $ln(y) = x$; however, with the understanding that this is an inverse function, it is normal to write $y = ln(x)$ or $f(x) = \ln(x)$.

### Part B: Exploration

In 1949, Willard Libby and a team of scientists at the University of Chicago discovered that the age of an organism could be found based on the amount of radioactive carbon it contains — a process known as *carbon dating*. Every object contains two types of carbon; radioactive carbon, carbon-14, and non-radioactive carbon, carbon-12. In living objects, the ratio of these two carbons is fixed, i.e., the amount of each carbon in living things remains the same. When a living organism dies, the carbon-14 is no longer replenished and starts to decay. The process of carbon dating aids scientists in determining the amount of time that has passed since an organism was alive. To calculate how long ago the carbon-14 stopped being replenished, scientists use logarithms. Let's investigate how logarithms are used.

Radioactive elements, such as carbon-14, have a specific rate of decay. The amount of time it takes for half of the radioactive component in the element to decay is known as the element's half-life, $h$. If $N(t)$ is the amount of radioactive material as a function of time, $t$, and if $N_0$ is the amount of radioactive material that was originally in the sample, and $t$ is the amount of time passed since death, then the amount of radioactive material remaining in the sample after time, $t$, is represented by $N(t)$, where

$$N(t) = N_0 e^{-\frac{t \ln 2}{h}}.$$

**3.** According to the definition of *half-life*, half of the original amount of the radioactive element should remain after one half-life has passed. Use the equation provided for $N(t)$ and set the time passed equal to one half-life $(t = h)$ to illustrate this relationship.

**4.** Suppose you know what $N,\ N_0$ and the half-life $h$ of an element is, but you would like to know the time that has passed since a specimen was alive. Describe a procedure to solve for $t$. Accompany your procedure with the algebraic manipulation of $N(t)$ to isolate $t$.

The equation you found in Exercise 4 is the general equation scientists utilize to determine the amount of time a specimen has been dead when they know the current and original amounts of a decaying element.

**5.** Use the fact that the half-life of carbon-14 is about 5700 years to rewrite the general equation you found in Exercise 4. This is the specific equation used for carbon dating.

**6.** Scientists find a wooden spoon and they want to use carbon dating to figure out how old it is. If the amount of carbon left is 85% of its original amount, how old is this artifact? (Note here that we are actually finding the age of the wood from which the spoon was made.)

**Part C: Graphing**

For the following functions: sketch a graph, identify the domain and range, and identify any critical values.

**7.** $f(x) = \log(x)$

**8.** $f(x) = \ln(x-2)$

**9.** $f(x) = \log_2(-x)$

**10.** $f(x) = -\ln(x+2)$

### Historical note:

The theory behind logarithms was developed by John Napier (1550-1617), who spent twenty years working on this topic. He published his work in a book entitled *Mirifici Logarithmorum Canonis Descriptio*. Logarithms were highly acclaimed at the time of their discovery because their properties allowed multiplication problems to be turned into addition problems.

Napier also introduced an early form of the calculator that involved lining up wooden rods or bones, but it was  difficult to use. His device was called Napier's rods or Napier's bones.

## Lesson 3.3: Growth and Decay

For each problem in the Exploration that follows, write an exponential function in the form $f(t) = a \cdot b^{kt}$, where $k$ is a constant, to describe the situation. For parts **a-d**: identify as exponential growth or exponential decay, domain, range, and sketch a graph.

### Exploration 3.3.1: An Investigation of Growth and Decay Models

| | |
|---|---|
| **a.** Write a function $F(t)$ that describes the value after $t$ years of a $\$2000$ investment that increases by $9\%$ per year. | **b.** Write a function $V(t)$ that describes the value after $t$ years of a $\$25{,}000$ car that depreciates (loses value) at a rate of $15\%$ per year. |
| **c.** Write a function $P(t)$ that describes the population after $t$ days of a colony of $1000$ fire ants that doubles in population every $13$ days. | **d.** Write a function $m(t)$ that describes the remaining mass after $t$ years of a $100$-gram sample of a radioactive compound with a half-life of $31$ years. |
| **e.** For your function in Part **a** above, find the value of your investment after $10$ years. | **f.** For the function in Part **a**, solve to find how long it would take to double your original investment. Try to solve this equation without using a graph or table. What problem do you encounter? |
| **g.** For the function in Part **c**, solve to find how long it would take for the ant population to reach 10,000. | **h.** Simplify the function from Exploration 3.2.1, $N(t) = N_0 e^{-\frac{t \ln 2}{h}}$, to show that the function can be written in the form $f(t) = a \cdot b^{kt}$. |

**Extension:** These three common exponential functions:

(1) $y = Ce^{kt}$,

(2) $y = y_0 + Ce^{kt}$, and

(3) $y = \dfrac{L}{1 + Ce^{-kLt}}$,

where $C, k, y_0$ and $L$ are constants, are commonly used to model population growth or decay situations.

**Note:** Each of these equations is a mathematical model for describing a physical process. Equation (1) represents simple growth and decay, Equation (2) is known as Newton's Law of Cooling, while Equation (3) is a general logistic model.

The standard *exponential growth model* as applied to, for example, population growth, is given by $P(t) = P_0 e^{kt}, k > 0$ where $P$ is the population present at time $t$ of an organism and $P_0$ is the initial value for the population of the organism under investigation at time $t = 0$. The constant $k$ is a growth constant inherent to a given population. This model assumes that the function grows without limit. However, it is often not the case that populations, for example, grow without limit.

In reality, population growth is often inhibited or limited by factors such as food or water supply, living space, predators, and so forth. In this case a different type of exponential function is used to model the situation, the *logistic growth model*. The logistic growth model is an exponential function that models situations where the growth of the independent variable is limited. The equation for this function is

$$P(t) = \frac{L}{1 + Ce^{-kLt}},$$ where $L$, $k$, and $C$ are constants with $k$ and $L > 0$.

The *logistic growth model* is quite important in population modeling and has application in other branches of mathematics such as chaos and dynamics. As a population model, the constant $L$ is called the *carrying capacity* of the model, and the line $y = L$ is a horizontal asymptote for the function.

### Exploration 3.3.2: The Logistic Growth Model

**PROBLEM:** Fruit flies are situated in a small glass bottle containing a limited amount of food. Suppose the fruit fly population after *t* days is given by the function

$$P(t) = \frac{230}{1 + 56.5e^{-(.0016)(230)t}}$$

**1.** How many fruit flies were originally placed in the bottle?

**2.** What is the carrying capacity of the small glass bottle as *t* gets larger (graph this function to help answer this question)?

**3.** When will the population of fruit flies be 200?

**4.** Using a graphing utility, change the various constants in the fruit fly equation one at a time and notice how each affects the characteristic "S" shape of the logistic function graph.

## Lesson 3.4: Using Functions Defined by Patterns in Application

Now it is time to explore how function patterns may be used in application. We will start with a fairly "well behaved" situation in that it is not hard to discern patterns within the data provided. Our goal in a future Exploration will be to use function patterns and other learned techniques to investigate data for which finding a model is not so obvious.

### Exploration 3.4.1: An Application of Functions Patterns

A chemical "marker," is used to trace metabolic activity in the heart. This substance has a half-life of about $3$ hours. Suppose a dose of this marker was injected into a patient. Let $M(t)$ be the amount of the marker measured in $mCi$ that remains over time, $t$, in hours, as shown in the table.

| $t$ in hours | $M(t)$ in $mCi$ |
|---|---|
| 3 | 5 |
| 6 | 2.5 |
| 9 | 1.25 |

1. Determine the number of $mCi$ that remain after $15$ hours.

2. Use function pattern properties that you learned previously to make a conjecture as to the type of function that models the given data. What type of function models this pattern?

3. Why can't you use the pattern to find $M(22)$?

4. Find a particular equation for $M(t)$ (leave your equation exact) and verify that all of the $M(t)$ values in the given table satisfy the equation.

5. Use your equation to calculate $M(22)$.

6. If another group presents a different equation that works for the given data, show that, in fact, the different equation is an equivalent form of your conjectured equation. Otherwise, can you use concepts learned in a previous Exploration to find a model for the radioactive marker data?