UNIT 2

Algebra and Geometry

By "algebra and geometry" the authors are referring to the fact that algebra can be viewed as an abstraction of geometry. Algebra is a language with equations as sentences. Geometry is developed based upon a logical expansion of definitions and axioms or first principles. While the topic of "algebra and geometry" is a common thread throughout this text, we will focus on one main aspect of this topic in this Unit. Thanks to René Descartes, the abstraction of algebra can be developed directly from the geometry of the plane when geometric figures are viewed as collections of identifiable points in the Cartesian or rectangular coordinate system.

Lesson 2.1: Algebra and Geometry

!! For the next Exploration, recall that the phrase *transformations in the plane* refers to translations, rotations, reflections, and dilations of figures in the plane.

Exploration 2.1.1: Algebra and Geometry Meet

Geometric transformations applied to graphs display interesting interactions between algebra and geometry. In particular, we know that the parabola that is the graph of

$$y = x^2$$

can be transformed into any other parabola of the form

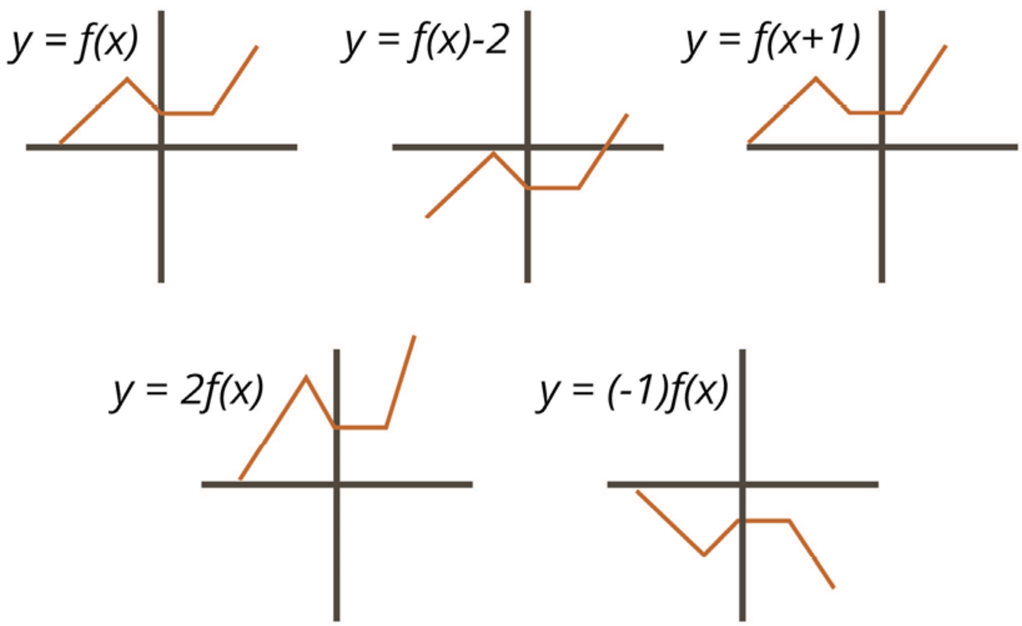
$$y = Ax^2 + Bx + C = a(x - h)^2 + k$$

using translations (horizontal and vertical), reflections, and/or dilations and contractions. Remember that h, k, and a can represent any real number, and the vertex of the parabola for

$$y = a(x - h)^2 + k$$

is (h, k).

The graph of $y = f(x)$ can be transformed in many ways. The first graph below is referred to as the "parent function," while the others represent transformations of this graph and are labeled with the algebraic notation for these transformations. In particular, the second graph is a vertical translation (2 units down), the third is a horizontal translation (1 unit left), the fourth is a dilation using a *scale factor of* 2, and the fifth is a reflection through the x-axis.



Part 1

1. Suppose that the graph of $y = x^2$ is transformed by a horizontal translation of h units. Find the roots of the resulting parabolas. [**Note:** Be sure to consider both cases $(h < 0$ and $h > 0)$ of the translation.]

2. Suppose that the graph of $y = x^2$ is transformed by a vertical translation of k units. Find the roots of the resulting parabolas.

3. Suppose that the graph of $y = x^2$ is transformed by a dilation using a scale factor. $a > 1$ Find the roots of the resulting parabolas.

4. Suppose that the graph of $y = x^2$ is transformed by the composition of a dilation using a scale factor $a > 1$, and a vertical translation of k units. Find the roots of the resulting parabolas.

5. Suppose that the graph of $y = x^2$ is transformed by a composition of a dilation using a scale factor $a > 1$, a vertical translation of k units, and a horizontal translation of h units. Find the roots of the resulting parabolas.

6. Show how this final answer relates to the quadratic formula.

Part 2

7. For the parent function $y = x^2$, compare this function to the ones below. State the type of transformations involved and how they have affected the parent function.

a. $y = -x^2$

b. $y = (x+2)^2$

c. $y = (x-3)^2$

d. $y = x^2 + 5$

e. $y = x^2 - 4$

f. $y = 2x^2$

g. $y = \frac{1}{3}x^2$

8. State the parent function present and then sketch the given function (without a calculator) based on the transformations applied to the parent function.

a. $y = -3|x-1| + 1$

b. $y = 2\sqrt{x+3} - 4$

Lesson 2.2: Complex Geometry and Roots

The quadratic function

$$f(x) = ax^2 + bx + c$$

can certainly be said to be everywhere present in secondary mathematics curriculum. Due to this fact, we will explore some of the interesting properties of this function throughout this text.

In the next Exploration, we will try to devise a way to visualize the location of the roots of a quadratic when it is the case that the roots are complex. In order to accomplish this task, we will allow the domain of the function to be the complex numbers. It is assumed that you are comfortable working with complex numbers and performing some basic operations with them.

!! Your instructor might choose to perform a rudimentary review of properties of complex number addition and multiplication along with an illustration of numbers in the complex plane.

Exploration 2.2.1: Complex Roots Visualization

While a graphing calculator is not recommended for use in graphing the quadratic in this Exploration, a calculator may be used to find the function values for the complex domain values under consideration.

Consider the graph of the quadratic

$$f(x) = x^2 + 4x + 7.$$

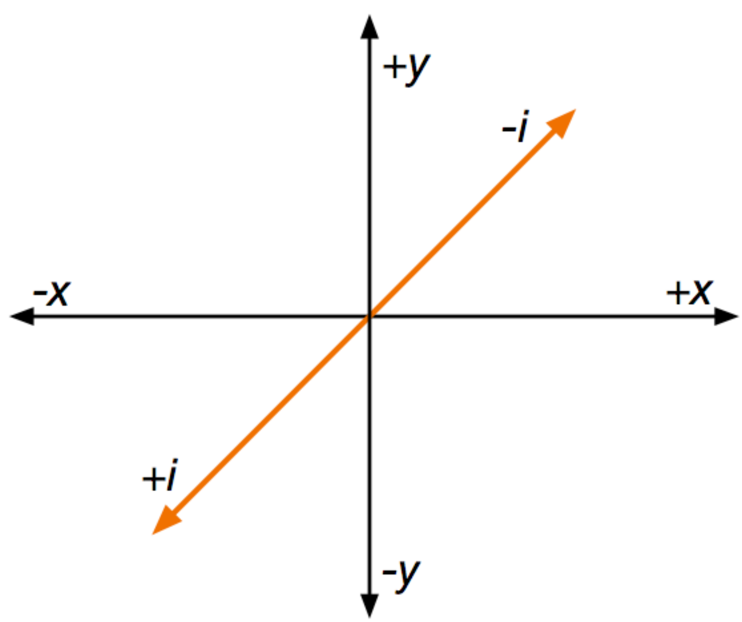
1. Graph $f(x)$ in detail in the xy plane of the coordinate system provided on the next page without using a graphing calculator. Verify that the zeros of the function are complex.

2. Suppose that we can use the complex numbers as the domain for $f(x)$. Do enough trials to find the values of a and b for which $f(a+bi)$ is a real number. Then, make a conjecture about your findings and prove it.

[Challenge: For $f(x) = ax^2 + bx + c$, find $d + ei$ that will generate real values.]

3. Lastly, can you visualize and draw a graphical representation of what your above answers imply about the real-valued outputs of f with regard to the inclusion of a complex domain? Try to do this using the 3-dimensional coordinate system provided below.

Axes template: (be sure to scale axes):



Exploration 2.2.2: Quadratic Formula

1. Based on your previous experience and the Exploration that you completed involving "roots" and complex numbers, state four ways one might find the roots (or zeros) of a quadratic function.

2. Now, using the general formula $ax^2 + bx + c = 0$, prove that one method that you have mentioned is really a general case of the other (i.e., derive the quadratic formula).

Exploration 2.2.3: The $P(x)$ Problem [A concept connections challenge]

Given integers $a, b,$ and c such that $0 < a < b$, and given

$$P(x) = x(x - a)(x - b) - 13$$

where $P(x)$ is divisible by $(x - c)$.

1. Find $a, b,$ and c.

2. Is your solution unique? Justify your answer.

Lesson 2.3: Conic Sections

We will now turn our attention to the *conic sections*. A conic section can be defined as the curve of intersection of a plane with a right circular cone. The curves created by these intersections are the circle, the ellipse, the hyperbola, and the parabola. There are also *degenerate* cases of these curves created, such as the point, the lines, or two intersecting lines. Associated with each of these conics is an analytic definition. As was mentioned previously, the curve generated by the quadratic equation is a parabola. One might ask, for example, how it is that the conic section, the analytic definition, and quadratic function describing a parabola are all related. This is the topic to be explored in this section in that it is another beautiful example of the interface and the meeting of algebra and geometry.

First we will look at the conic sections and then their associated analytic definitions. We visualize each conic section as the intersection of a plane with a right circular cone (see Figures 2.3-1, 2.3-2, and 2.3-3).

Figure 2.3-1: Parabola.

Figure 2.3-2: Ellipse.

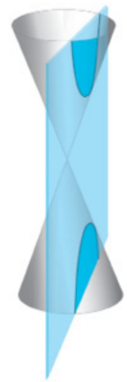


Figure 2.3-3: Hyperbola.

Depending on the orientation of the plane, the resulting curve can be a parabola, ellipse, circle, or hyperbola (with degenerate forms: point and line or lines). The reader is asked to describe the situation in which a ***circle*** arises.

The analytic definitions of the conic sections are:

DEFINITION: Parabola. We define a parabola (see Figure 2.3-4) to be the set of all points in the plane that are equidistant from a fixed point and a fixed line. The fixed point is called the focus of the parabola and the fixed line is called the *directrix* of the parabola. A special point on the parabola is the *vertex*, the midpoint of the perpendicular segment from the focus to the directrix.

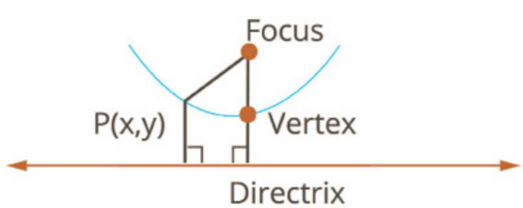


Figure 2.3-4

DEFINITION: Ellipse. We define an ellipse to be the set of all points in the plane for which the sum of the distances to two fixed points is constant. The fixed points are called the *foci* of the ellipse. This definition is illustrated in Figure 2.3-5. We define the center of an ellipse to be the midpoint of the line segment connecting the foci.

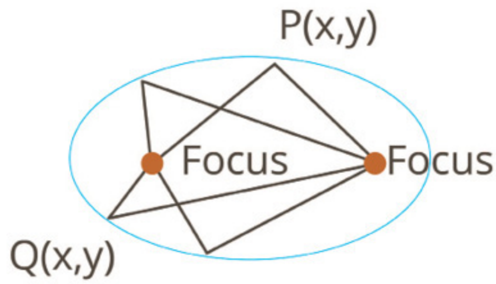


Figure 2.3-5

DEFINITION: Hyperbola. We define a hyperbola to be the set of all points in the plane such that the difference of the distances between two fixed points is a constant. This definition is illustrated in Figure 2.3-6.

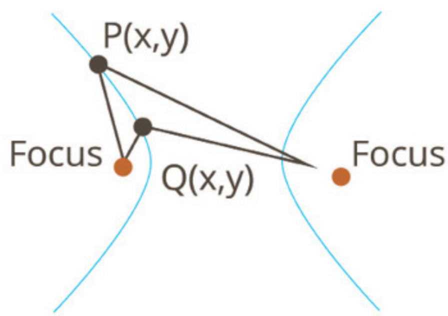


Figure 2.3-6

EXERCISE: Based on your discussions in Exploration 0.1.3, can you provide an analytic definition for the circle?

Exploration 2.3.1: The Conics — Equations from Definitions

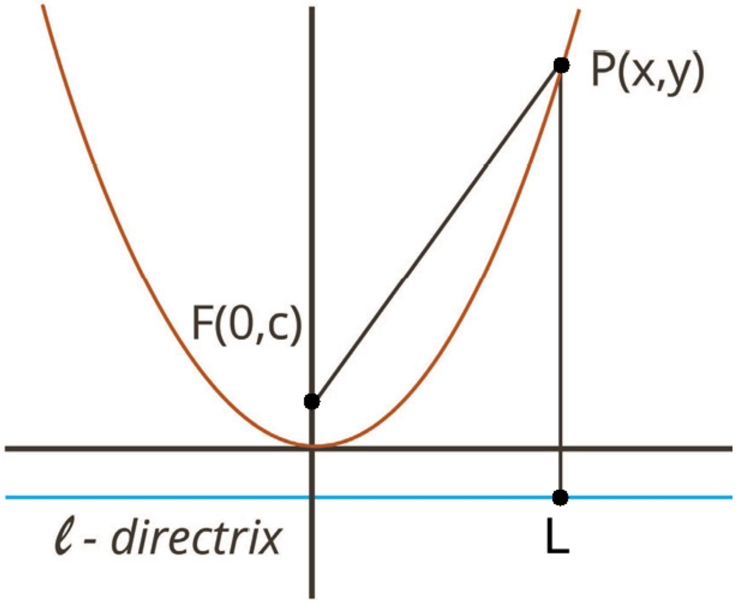


Figure 2.3.1-1

A. Parabola

Given: A fixed point F and a fixed line l. The parabola consists of all points P such that $\overline{FP}$ is equal to $\overline{PL}$. (The distance from F to P is the same as the distance from P to the line l; in symbols $\overline{FP} = \overline{PL}$.)

1. Without loss of generality, orient the parabola so that the point F (called the *focus of the parabola*) is on the y-axis and the line l (called the *directrix of the parabola*) is parallel to the x-axis.
2. From Figure 2.3.1-1, let P be a general point on the parabola. What are the coordinates of P?
3. Use the distance formula to find $\overline{FP}$.
4. Use the distance formula to find $\overline{PL}$.
5. Substitute the expressions in 3 & 4 in the equation $\overline{FP} = \overline{PL}$, and simplify using algebraic techniques in order to derive a general equation that models the parabola.

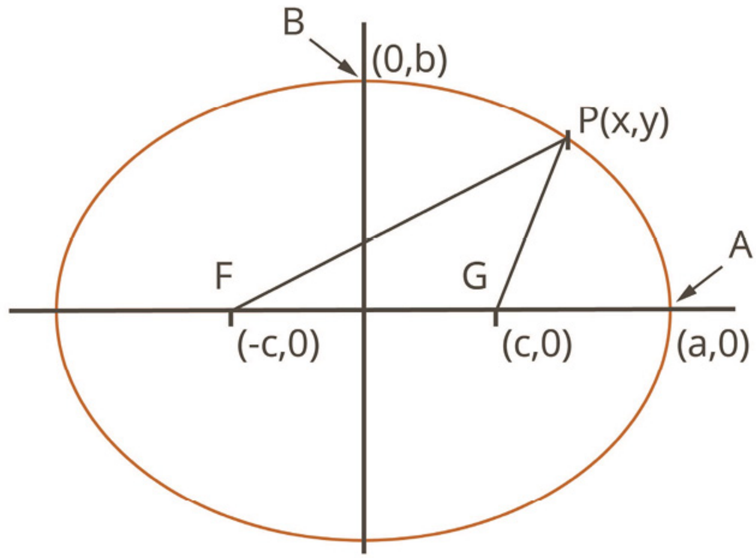


Figure 2.3.1-2

B: Ellipse

Given: Two fixed points F, G and *a fixed positive number* k. The ellipse consists of all points P such that $\overline{FP}+\overline{GP}=k$.

The fixed points F and G have coordinates $(-c,\,0)$ and $(c,\,0)$, respectively. The points A and B are points where the ellipse intersects the positive x-axis and positive y-axis, respectively.

1. Use the distance formula to express the relationship $\overline{FP}+\overline{GP}=k$ in terms of the coordinates of an arbitrary point $P(x,\,y)$, as pictured, which lies on the ellipse. (Your expression should involve a sum of two square roots.)
2. Use algebraic techniques to eliminate the square roots that occur in 1. (Note: This involves squaring twice; it helps to simplify the result obtained by squaring the first time before squaring a second time.) Your expression should involve x, y, c, and k.
3. **Verify** that $k = 2a$ and $c^2 = a^2 - b^2$, knowing that *A* and *B* are points on the ellipse.
4. Substitute the values for k and c into your derived equation to obtain the standard equation of the ellipse centered at the origin with semi-major axis of length a and semi-minor axis of length b:

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

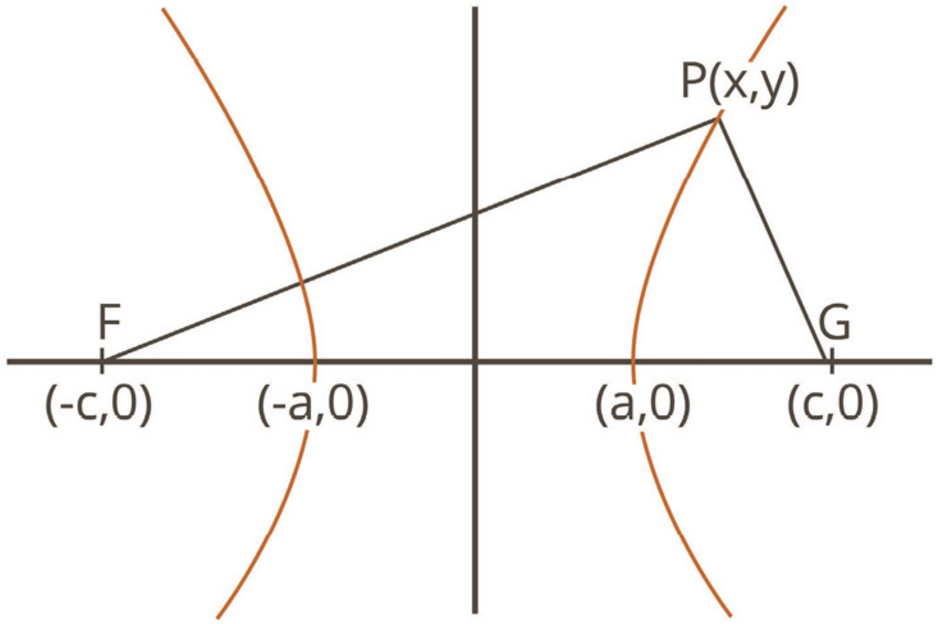


Figure 2.3.1-3

C: Hyperbola:

Given: Two fixed points F, G and *a fixed positive number* k. The hyperbola consists of all points P such that $\left|FP - GP\right| = k$.

1. Follow the procedures outlined for the **Ellipse** noting that the substitution $b^2 = c^2 - a^2$ will lead to the standard form of the hyperbola centered at the origin.

Exploration 2.3.2: The Conics — Identifying Parameters

Part I. For the following problems convert the equation to standard form, create a sketch, and identify all of the critical parameters that apply.

1. $4x^2 - 8x + 9y^2 - 72y + 112 = 0$

2. $-4x^2 + 40x + 25y^2 - 100y + 100 = 0$

3. $x^2 - 4x - 24y + 28 = 0$

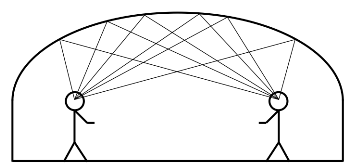
4. $4x^2 - 24x - 36y^2 - 360y + 864 = 0$

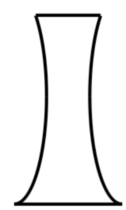
5. $3y^2 - 4x - 6y + 23 = 0$

6. $16x^2 + 16x + 9y^2 + 72y + 4 = 0$

Part II. For the following problems apply your knowledge about Conics to answer the questions.

7. Two people enter a whispering gallery and want to know where they should stand so they can hear the other person's whisper. The gallery is 100 feet wide and the vertical distance from their head to the peak of the ceiling is 30 ft.





8. A hyperbolic cooling tower is 100 meters wide at the base. The most narrow section of the tower is 100 m above the ground and 40 m wide. How wide is the cooling tower at its highest point, which is 160 m above the ground?

9. A flashlight has a parabolic reflector that has a diameter of 52.25mm a depth of 23.95mm. Where should the bulb be placed for so the flashlight is optimized for a focused beam of light?

Part III. Now consider, $0 = Ax^2 + Cy^2 + Dx + Ey + F$. This is the general form of the equation for any of the conic sections (neglecting rotations).

10. How would one know which conic is represented by a given equation in this form?

11. **[Extension]** Put each of the conics represented in general form into the standard form for the specific conic?

Functions as Models: Matrix and Regression Method

At this point our attention will be turned toward the concept of using functions to *model* discrete data. Mathematics modeling is a process in which a description of a system or data associated with a system is developed using mathematical language. This will be accomplished using either matrices or statistical regression to find functions that best model the data under investigation.

Lesson 2.4: Using Matrices to Find Models

!! Your instructor might choose to perform a rudimentary review of properties of matrix addition and multiplication. In this section, we will use matrices to solve systems of equations. The use of a calculator to solve some of the systems is suggested but not required.

Suppose we have a 3 x 3 matrix A that represents a system of equations along with a 3 x 1 solution matrix B. In this system, we are trying to solve for three unknown coefficients a, b, and c represented by the 3 x 1 matrix C. One might represent this situation with a matrix equation such as

$$[A]\,[C] = [B]\,.$$

Keep in mind that our goal is to find the values of the coefficients of the matrix C. Recall that the way to accomplish this task is to multiply both sides of the equation by the inverse of the matrix A (provided that it exists) to yield

$$[A^{-1}]\,[A]\,[C] = [A^{-1}]\,[B],$$

where $[A^{-1}]\,[A]$ is equal to the identity matrix. This simplifies to

$$[C] = [A^{-1}]\,[B].$$

While it is clear that this process allows one to find the solution C, one should ask, "Given an invertible system representing A, how does one find the inverse of A?" The next example sheds light on the answer to this question.

EXAMPLE: How to Find the Inverse of a 3×3 Matrix

The goal is to find A^{-1} if

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 3 \\ -1 & 1 & 1 \end{bmatrix}$$

Begin by forming the following 3 × 6 augmented matrix. Perform matrix row operations to obtain the identity matrix on the left side, and perform the same operations on the right side of this matrix.

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 2 & 1 & 3 & 0 & 1 & 0 \\ -1 & 1 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\begin{array}{r} \\ R_2 - 2R_1 \rightarrow \\ R_3 + R_1 \rightarrow \end{array} \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & -2 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 1 \end{array}\right]$$

$$\begin{array}{r} \\ \\ R_3 - R_2 \rightarrow \end{array} \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & -2 & 1 & 0 \\ 0 & 0 & 1 & 3 & -1 & 1 \end{array}\right]$$

$$\begin{array}{r} R_1 - R_3 \rightarrow \\ R_2 - R_3 \rightarrow \\ \\ \end{array} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -2 & 1 & -1 \\ 0 & 1 & 0 & -5 & 2 & -1 \\ 0 & 0 & 1 & 3 & -1 & 1 \end{array}\right]$$

The right side is equal to A^{-1}. That is

$$A^{-1} = \begin{bmatrix} -2 & 1 & -1 \\ -5 & 2 & -1 \\ 3 & -1 & 1 \end{bmatrix}$$

It can be verified that $A^{-1}A = I_3 = AA^{-1}$.

Exploration 2.4.1: The Inverse of a Matrix

In this quick Exploration, you are asked to find the inverse of matrix A where:

$$A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 3 \\ 4 & -3 & 8 \end{bmatrix}$$

Of course, this should be done by hand. How might you check your answer?

Challenge: Based on this example, can you provide some general informal justification for why this process works?

NOTE:

If it is not possible to obtain the identity matrix on the left side of the augmented matrix by using matrix row operations, then A^{-1} does not exist.

In the next Exploration you will be asked to use what you have learned thus far in this section to find a model function or relation to fit data. This will be done in the context of working with two types of conics that have been previously discussed: the parabola and the circle.

Exploration 2.4.2: Using Matrices to Model Functions and Relations

1. How many data points does one need in order to use matrices to find an equation for a parabola of the form:
$$y = ax^2 + bx + c\ ?$$
2. Recalling that $[C] = [A^{-1}]\,[B]$, explain why this matrix method will help find the desired equation.
3. How many points does one need to use matrices to find an equation for a circle of the form:
$$0 = x^2 + y^2 + Dx + Ey + F\ ?$$
4. Is the above situation a function or a relation? Why?
5. Find the equation for a circle containing the points:
$(0, 0), (0, 5), (3, 3)$.
6. Find the equation for a circle containing the points:
$(-1, -3), (-2, 4), (2, 1)$.
7. Find the equation for a parabola containing the points:
$(0, 0), (20, 47), (30, 88)$.
8. Find the equation for a parabola containing the points:
$(0, 1.60), (10, 1.85), (20, 2.00)$.
9. Lastly, put all equations found above into standard form.

Lesson 2.5: Using Statistical Regression to Fit a Function to Bi-variate Data

A regression method called *the method of least squares* will be used in this section to fit a *best fit* linear function to data that is conjectured to be linear. A simple linear regression is accomplished by finding a line that minimizes the distance between actual data points and the predicted values $\hat{y}$ on the best fit line. The difference between the predicted value at the i^{th} data point $\hat{y}_i$ and the observed value y_i is called a *residual*, symbolized e^i, such that

$$e_i = y_i - \hat{y}_i.$$

The method of least squares used to find the predicted regression line employs an optimization technique from calculus to minimize the sums of the squares of the *deviations* $y_i - \hat{y}_i$ such that an estimated or fitted regression line of the form

$$\hat{y} = bx + a$$

can be produced.

An example of a scatter plot of data points and a best fit regression line is displayed in Figure 2.5-1.

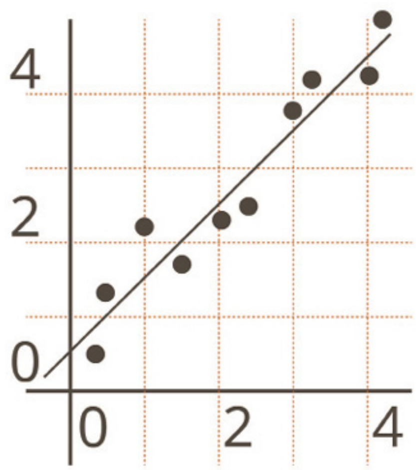


Figure 2.5-1 – A Scatter Plot and the Regression "Best Fit" Line

!! One can use technology to avoid some of the tedious mathematics involved in producing the regression equation of the best fit line. Graphing calculators, statistical software, or spreadsheet programs such as Microsoft Excel will calculate a linear regression equation to fit data.

Exploration 2.5.1: An Example of Linear Regression - Thunderstorms

This Exploration should be done as a full-class exercise as a way for your instructor to teach you how to perform a linear regression investigation on relevant technology.

It is conjectured that in a lightning storm, the distance between a person and the lightning is linearly related to the time interval between the flash and the bang. Consider d the distance to the storm in kilometers as a function of time t in seconds. Suppose, as an experiment, a friend travels along with the storm and reports the actual distance that the storm is from your house as you record the seconds between the flash and the bang.

1. Make a scatter plot of the data and use linear regression to write the particular equation for this direct variation function.

t	2.9	6.1	14.9	28.9	37.2
d	1.0	2.0	5.0	10.0	12.0

2. Use your model to work backwards in order to calculate the times for the thunder sound to reach you from lightning bolts that are 1.5, 2.5, and 15 kilometers away. What do you call the processes of looking within and beyond your actual data?

CHALLENGE: What would your linear equation be with seconds and *miles* as your units?

Exploration 2.5.2: An Example of Linear Regression – Charles's Law

In this Exploration, practice the use of relevant technology to perform the linear regression process.

Physicist Jacques Charles (1746-1823) discovered that the volume of a gas at a constant pressure is linearly dependent on the temperature of the gas. The table below illustrates this relationship. In the table, hydrogen is held at a constant pressure of one atmosphere. The volume V is measured in liters and the temperature T is measured in degrees Celsius.

T	–40	–20	0	20	40	60	80
V	19.15	20.79	22.43	24.08	25.72	27.36	29.00

1. Consider V as a function of T and make a scatter plot of the data.

2. Use the table above and what you have learned about linear regression to find a model for the linear relationship.

Assuming that one started with 1 mole of gas at constant pressure, have you seen the values that you found for the T coefficient and the constant in the equation before? If not, you might consider doing some background research related to gas laws.

3. Solve the equation that you have found for T and substitute in values of V that get closer and closer to 0. What number for temperature do you seem to be approaching? Does this number look familiar? If so, what is this value?

In fact, through use of similar methods to that of the method of least squares, technology can be used to find regression equations to fit linear, quadratic, power, exponential, logarithmic models or other functions to supplied data. In this sense, theoretically, one could fit many different types of functions to a given set of data. So the question should be asked, "How, then, does one know which is the best type of model or function to fit to a sample of data points?"

Luckily, there are various "tools" available that help one decide which type of regression to use to fit a model or function to data. One tool that can be applied is the "functions defined by patterns" method explored in previous sections. Another tool available to us is the *correlation coefficient* associated with linear regression. The correlation coefficient r measures and describes a relationship or trend between two variables. Note that a correlation between two variables does not imply a causal relationship between the two variables. The relationship can, however, be characterized as having a positive correlation or a negative correlation depending on whether the linear model associated with the data is increasing or decreasing respectively. The correlation coefficient r can take on the values

$$-1 \leq r \leq 1$$

The value for r is obtained by computing the ratio of a measure of covariability between variables divided by a measure of the product of variations within data associated with each of the two variables under investigation. The actual derivation of the formula for r is beyond the scope of this course. If the value of r is close to -1, this indicates a strong linear correlation between variables from data with a negative trend. If r is close to +1, a strong linear correlation between variables from data with a positive trend is indicated. A value of r close to zero indicates little linear correlation between the variables under investigation.

It must be noted that most graphing calculators and statistical and spreadsheet programs are programmed to display the value of the correlation coefficient r when one performs a statistical regression. However, other types of correlation between variables are also reported depending on the type of regression performed (i.e., linear versus other types of function regression). These are summarized here. Your instructor may choose to discuss these coefficient values at length with you.

r^2 - **The Coefficient of Determination**

The value r^2, the *coefficient of determination*, measures the proportion of total variation in the values of Y that can be accounted for or explained by a linear relationship with the values of the random variable X.

R^2 - **The Coefficient of Multiple Determination**

Both graphing calculators and software programs report an R^2 value for some non-linear regression models. This value is called the *coefficient of multiple determination*. It is analogous to the *coefficient of determination,* r^2, which applies strictly to linear situations. R^2 can be thought of as a multiple linear least squares fit.

For example, if $Y = ax^2 + bx + c$, then Y can be considered *linear* in the coefficients a, b, and c with $Y = aX_2 + bX_1 + c$ where $X_2 = (X_1)^2$. This same argument can be used for higher-degree polynomials.

This quantity represents the proportion of the total variation in the response Y that is explained by the fitted model. The closer to 1 the R^2 value is the better the 'fit' of the chosen regression curve.

Lastly, be aware that an r value (correlation coefficient) may be reported by graphing calculators and software programs for non-linear functions such as logarithmic, exponential, and power functions that can be made *linear by transformation*.

Another tool available to you that can help justify the choice of function used to fit data is the *residual plot*. Recall that a residual e_i is the difference between the predicted value at the i^{th} data point $\hat{y}_i$ and the observed value y_i. For any type of regression chosen, one can plot the residual corresponding to each data point as a function of the explanatory or generating value of the bivariate data. If the correct regression model has been chosen, one would expect the residual plot to consist of a scatter plot of values nicely distributed above and below the value $e = 0$ as pictured in Figure 2.5-2.

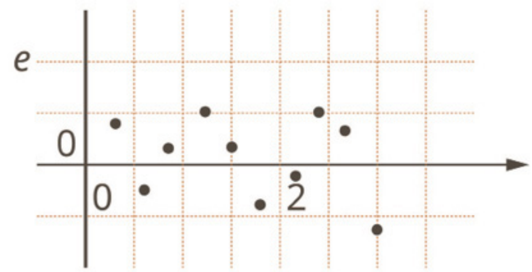


Figure 2.5-2 – A Residual Plot

On the other hand, if the regression model that you have chosen to fit your data is not truly the 'best fit' equation, the residual plot associated with the predicted and actual data will display a definite pattern as is depicted in Figure 2.5-3. Why do you suppose this is the case?

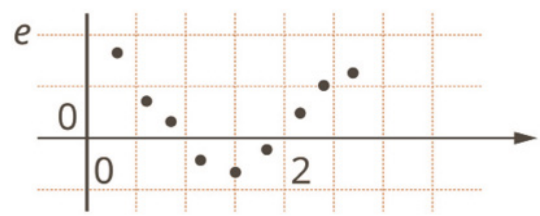


Figure 2.5-3 – A Residual Plot Displaying a Pattern

One must be aware of the fact that there is a limitation to using the residual plot as a tool for choosing which type of regression model is best to apply to a data set. The limitation is that if one does not have enough data points in the set, then it is hard to tell if there is truly a pattern in the associated residual plot.

Exploration 2.5.3: An Application Activity Using Residuals

Average Weight and Length Measurements for a Selected Species of Fish

Age in years	Length in centimeters	Weight in grams
1	5.2	2
2	8.5	8
3	11.5	21
4	14.3	38
5	16.8	69
6	19.2	117
7	21.3	148
8	23.3	190
9	25.0	264
10	26.7	293
11	28.2	318
12	29.6	371
13	30.8	455
14	32.0	504
15	33.0	518
16	34.0	537
17	34.9	651
18	36.4	719
19	37.1	726
20	37.7	810

1. Our conjecture is that this is an exponential situation. Perform an exponential regression using weight in grams as a function of length in centimeters. Draw a graph of the scatter plot for the data that includes your regression curve.

2. (Optional Extension) Next, construct and draw a graph of the residual plot for the regression to confirm or reject our conjecture. Comment on the results.

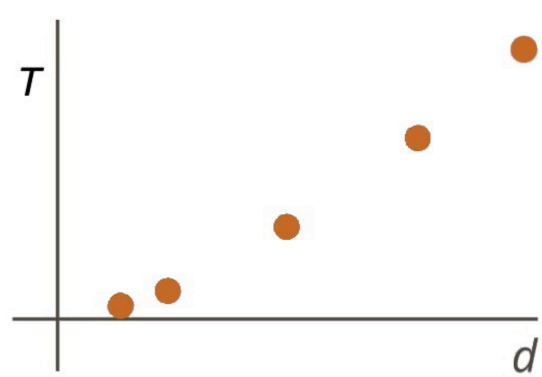
3. Might there be a better model for this data? Test your answer to this question.

The next Exploration is an activity that will synthesize some of the various techniques used to find a function or model that best fits a data set.

Exploration 2.5.4: A Planets Exploration - Using "Real" Data

The table below shows the periods of the orbits of each planet in years and the mean (half the sum of the greatest and smallest distances or the semi-major axis) distance from the Sun in kilometers.

Name	Period - *T* (yrs)	Millions of km from the Sun (semi-major axis-distance - *d*)
Mercury	0.24	57.9
Venus	0.61	108.2
Earth	1	149.6
Mars	1.88	228.0
Jupiter	11.86	778.5
Saturn	29.46	1433.3
Uranus	84.01	2872.6
Neptune	164.79	4493.6



1. Use "function patterns" (remember this is real data) to try to decide what type of function this data represents considering *period, T*, as a function of mean *distance, d*, from the Sun (the semi-major axis of the planet's orbit).

2. Based on your answer to (1), do a regression to find the equation for the function that fits this data. In addition, comment on the r or R^2 value (whichever is appropriate based on your chosen model) associated with the equation.

3. Plot the scatter plot as shown above along with the regression curve to see how well the regression equation fits the data.

4. Kepler derived his three laws of planetary motion from analysis of data such as that in the table above. Research *Kepler's Third Law* in a reference text or in a physics text. Does your regression equation agree with that law?

5. (Optional Extension) Now explore the data and regression equation using residuals. Does the residual plot support your choice for the type of regression that you chose? Are there problems with trying to interpret the residual plot for this Exploration?