12.1: Sequences

- Sequences and Summation Notation

Sequences:

A sequence is a set of numbers written in a specific order: $a_1, a_2, a_3, \dots, a_n, \dots$
The number a_1 is called the *first term*, a_2 is the *second term*, and in general a_n is called the *nth term*.
Since for every natural number *n* there is a corresponding number a_n we define a sequence as a function.

EX: $f(1), f(2), f(3), f(4)$ or $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}$

Definition of a Sequence:

A sequence is a function *f* whose domain is the set of natural numbers. The values $f(1), f(2), f(3), \dots$ are called terms of the sequence.

Example 1: Find the first five terms and the 100th term of the sequence defined by each formula.

a) $a_n = 3n+1$

b) $b_n = \frac{n-1}{n}$

c) $c_n = (-1)^n$

Example 2: Find the nth term of a sequence whose first several terms are given.

d) $5, 8, 11, 14, \dots$

e) $-3, 6, -12, 24, \dots$

Recursively Defined Sequences:

Some sequences do not have simple defining formulas like the preceding examples. The nth term of a sequence may depend on previous terms. A sequence defined this way is called **recursive.**

Example 3: Find the first five terms of the following recursive sequences.

f) $a_n = 2a_{n-1}$ and $a_1 = 2$

g) $b_n = b_{n-2} + 2b_{n-1}$ and $b_1 = 0, b_2 = 3$

The Partial Sums of a Sequence:

For the sequence $a_1, a_2, a_3, \ldots, a_n, \ldots$ the **partial sums** are

$$S_1 = a_1$$
$$S_2 = a_1 + a_2$$
$$S_3 = a_1 + a_2 + a_3$$
$$S_4 = a_1 + a_2 + a_3 + a_4$$
$$\vdots$$
$$S_n = a_1 + a_2 + a_3 + \ldots + a_n$$
$$\vdots$$

S_1 is called the **first partial sum**, S_2 is the **second partial sum**, and S_n is called the ***n*th partial sum**.

The sequence $S_1, S_2, S_3, \ldots, S_n, \ldots$ is called the **sequence of partial sums.**

Example 4: Find the first four partial sums and the *n*th partial sum of the sequence defined below.

h) $a_n = \frac{2}{3^n}$

i) $a_n = \sqrt{n} - \sqrt{n+1}$

Sigma Notation:

Given a sequence $a_1, a_2, a_3, \ldots, a_n, \ldots$ we can write the sum of the first *n* terms using **summation notation** or **sigma notation.** The notation derives its name from the Greek letter Σ (capital sigma or S)

$$\sum_{k=1}^{n} a_k = a_1 + a_2 + a_3 + \ldots + a_n$$

Its is read "The sum of a_k from $k = 1$ to $k = n$." The letter *k* is called the index.

Example 5: Find each sum.

j) $\sum_{k=1}^{4} (2k - 1)$

k) $\sum_{j=2}^{5} j^2 - 4$

l) $\sum_{i=1}^{50} (-3)$

Example 6: Write each sum using sigma notation.

m) $1 + 2 + 3 + 4 + \ldots + 100$

n) $1 + 4 + 9 + \ldots + 100$

Arithmetic Sequence:

An arithmetic sequence is a sequence of the form: $a,\ a+d,\ a+2d,\ a+3d,\ \ldots$

The number a is the **first term** and d is the **common difference** of the sequence.

The ***n*th term** of an arithmetic sequence is given by: $a_n = a + (n-1)d$

The **partial sum** of an **arithmetic sequence** is given by: $S_n = \frac{n}{2}(a + a_n)$

Example 1: Find the *n*th term of the arithmetic sequence. What is the 100^{th} term? Find the sum of the first 20 terms.

a) **5, 8, 11, …** b) $a = 3,\ d = -4$

Example 2: The eighth term of the arithmetic sequence is 75. And the twentieth term is 39.

(a) Find the first term and the common difference.
(b) Give a recursive formula for the sequence.
(c) What is the *n*th term of the sequence?

Sum of the First *n* Terms of an Arithmetic Sequence

Let $\{a_n\}$ be an arithmetic sequence with first term a_1 and common difference d. The sum S_n of the first *n* terms of $\{a_n\}$ may be found in two ways:

$$S_n = a_1 + a_2 + a_3 + \ldots + a_n$$
$$= \Sigma\left[a_1 + (k-1)d\right] = \frac{n}{2}\left[2a_1 + (n-1)d\right]$$
$$S_n = a_1 + a_2 + a_3 + \ldots + a_n$$
$$= \Sigma\left[a_1 + (k-1)d\right] = \frac{n}{2}\left[a_1 + a_n\right]$$

Example 3: Find the *nth* term of the arithmetic sequence given the following information. Then find the sum of the first 15 terms.

c) $a_4 = 4,\ a_{30} = 108$

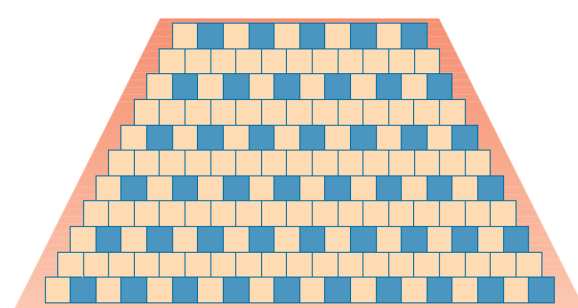
d) $a_4 = -3,\ S_{20} = -320$

Example 4: A partial sum of an arithmetic sequence is given. Find the sum.

e) $(-3)+(-1)+1+\ldots+35$

Example 5: An arithmetic sequence has $a = 6$ and a common difference of $d = 4$. How many terms of this sequence must be added to get 3870?

Example 6: A ceramic tile floor is designed in the shape of a trapezoid 20 feet wide at the base and 10 feet wide at the top. The tiles, 12 inches by 12 inches, are to be placed so that each successive row contains one less tile than the preceding row. How many tiles will be required?



Geometric Sequences:

A geometric sequence is a sequence of the form: where $a_1 = a$ $a_1, a_1r, a_1r^2, a_1r^3, \ldots$

Where the number a is the **first term** and r is the **common ratio** of the geometric sequence.

The ***n*th** term of the sequence is given by: $a_n = a_1r^{n-1}$.

The **partial sum** of a **geometric sequence** is given by: $S_n = a_1\dfrac{1-r^n}{1-r}$ or $\sum_{k=1}^{n} a_1r^{k-1} = a_1\bullet\dfrac{1-r^n}{1-r}$, $r \neq 1$

A geometric sequence can be **defined recursively**, as $a_1 = a$, $\dfrac{a_n}{a_{n-1}} = r$ or $a_1 = a$, $a_n = ra_{n-1}$.

Series:

An expression of the form $S_n = a_1 + a_2 + a_3 + \cdots$ is called a series.

Example 1: Find the *n*th term of the geometric sequence. Then find the 10th term and the sum of the first ten terms.

a) $4, -8, 16, -32, \ldots$

b) $192, 48, 12, 3, \ldots$

Example 2: Find the *n*th term of the geometric sequence described below. Then find the 10th term and the sum of the first ten terms.

c) $a_4 = 48,\ a_9 = 1536$

Example 3: Find the *n*th term of the geometric sequence described below. Then find the 10th term and then find the recursive formula for each sequence.

d) $10, 9, \dfrac{81}{10}, \dfrac{729}{100}$

e). $3, 2, \dfrac{4}{3}, \dfrac{8}{9}$

<u>Sum and Convergence of an Infinite Geometric Series:</u>

If $|r| < 1$, then the infinite geometric series converges. Its sum is : $S = \frac{a_1}{1-r}$ or $\sum_{k=1}^{\infty} a_1 r^{k-1} = \frac{a_1}{1-r}$
If the series does not converge we call it **divergent**.

Example 4: Find the sum of the infinite geometric series if possible. State if they are converges or diverges.

f) $-12+6-3+\frac{3}{2}-\ \ldots$

g) $\frac{100}{9}+\frac{10}{3}+1+\ldots$

h) $10-25+\frac{125}{2}-\ldots$